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ON A CLASS OF REFLECTION GEOMETRIES

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ON A CLASS OF REFLECTION GEOMETRIES

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SUNTO. — In questa Nota si introduce un sistema di assiomi che permette di caratterizzare una particolare classe di *geometrie di riflessione*. Tali geometrie consistono di uno *spazio di incidenza* (detto anche *spazio lineare* o *spazio di rette*) *con parallelismo* dotato di un insieme transitivo di dilatazioni involutorie e di un insieme regolare di traslazioni. Esse risultano quindi anche rappresentabili mediante una opportuna classe di *cappi di incidenza*.

Introduction

By a *reflection* we usually understand an involutory movement in an absolute plane with a pointwise fixed line. As a matter of fact it is well known the possibility to define in an abstract way an absolute space by means of a group Γ provided with a set of involutory elements fulfilling suitable axioms. The group Γ is named *reflection group* (*Spiegelungsgruppe*) and *reflection geometry* (*Spiegelungsgeometrie*) is the geometric structure defined in this way (see e.g. [2], [4]).

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In this paper we shall be concerned with a different notion of "reflection" which could be thought as a generalization of the classical notion of point-reflection in an absolute plane. This notion of reflection is introduced in an abstract way by means of suitable axioms (A1)-(A4) stated for any arbitrary set $\mathcal{P}$ of elements, named *points* (cf. § 1). Our aim is to provide the point set $\mathcal{P}$ with a *line structure* $\mathcal{L}$ which could be consistent with the reflection axioms. This will be obtained by means of a new set of axioms: (D1)-(D3); in this way $(\mathcal{P}, \mathcal{L})$ turns out to be an *incidence space* (cf. e.g. [3], [4]) (*linear space* or *line space* in other Authors) and furthermore in $\mathcal{L}$ a parallelism relation can be defined (§ 2). The reflections defined by the axioms (A1)-(A4) give rise to a set of involutory dilatations and a transitive set of translations for the space $(\mathcal{P}, \mathcal{L}, /\!\!/)$ (§ 3). This allow us to provide the incidence space $(\mathcal{P}, \mathcal{L}, /\!\!/)$ whit a structure of *incidence loop with parallelism* which is unique up to isomorphisms (§ 4).

1. - Existence of reflections

Let $\mathcal{P}$ be a set of elements, which henceforth we shall call *points*. Let us assume for each point $a \in \mathcal{P}$ a bijection $\tilde{a} : \mathcal{P} \longrightarrow \mathcal{P};\ x \longrightarrow \tilde{a}(x)$ is defined such that the following axioms are fulfilled:

A1. $\forall x \in \mathcal{P} : \tilde{x}(x) = x;$

A2. $\forall\ a, b, x \in \mathcal{P},\ a \neq b \implies \tilde{a}(x) \neq \tilde{b}(x);$

A3. $\forall a \in \mathcal{P} : \tilde{a}^2 := \tilde{a} \circ \tilde{a} = id;$

A4. $\forall x, y \in \mathcal{P}\ \exists\, a \in \mathcal{P} : \tilde{a}(x) = y.$

Henceforth these bijections will be called *reflections*. In the following we shall always denote by "*id*" the identity map and by "$\circ$" the composition of mappings. Thus we have:

1.1. - *Let* a, b, x, y *be any points of* $\mathcal{P}$; *then:*

(i) $x \neq a \implies \tilde{a}(x) \neq x;$

(ii) *if for some* $z \in \mathcal{P}$ *it is* $\tilde{a}(z) = \tilde{b}(z)$, *then* $a = b$;

(iii) *if* $\tilde{a}(x) = y$, *then* $\tilde{a}(y) = x$.

PROOF. -

(i) By (A2) and (A1): $a \neq x \implies \tilde{a}(x) \neq \tilde{x}(x) = x$.

(ii) If $a \neq b$ by (A2) we have for any $x \in \mathcal{P}$ $\tilde{a}(x) \neq \tilde{b}(x)$ which is a contradiction.

(iii) Immediate by (A3). □

From (1.1, ii) it follows immediately:

1.2. - *For any $x, y \in \mathcal{P}$, the element $a \in \mathcal{P}$ which exists by* (A4) *is uniquely determined.*

If we denote $\tilde{\mathcal{P}} := \{\tilde{a} : a \in \mathcal{P}\}$, let us define:

$$\mathfrak{T} := \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} := \{\tilde{a} \circ \tilde{b} : a, b \in \mathcal{P}\} .$$

By (A3) $id \in \mathfrak{T}$. Then:

1.3. - (i) *For each* $\tau \in \mathfrak{T} \setminus \{id\}, \forall x \in \mathcal{P}$ *we have:* $\tau(x) \neq x$;

(ii) $\mathfrak{T}$ *acts transitively on* $\mathcal{P}$.

PROOF. - (i) Let us suppose $y \in \mathcal{P}$ such that $\tau(y) = y$. Then if we denote $\tau := \tilde{a} \circ \tilde{b}$, we have $\tilde{a} \circ \tilde{b}(y) = y$ which implies $\tilde{a}(y) = \tilde{b}(y)$, by (A3), and thus by (1.1, ii) $\tilde{a} = \tilde{b}$ i.e. $\tau = id$, which is a contradiction.

(ii) For any $x, y \in \mathcal{P}$, by (A1) and (A4) there exists $a \in \mathcal{P}$ such that $\tilde{a}(x) = \tilde{a} \circ \tilde{x}(x) = y$. □

1.4. - *The following conditions are equivalent:*

(i) $\mathfrak{T}$ *acts regularly on* $\mathcal{P}$;

(ii) $\tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \subseteq \tilde{\mathcal{P}}$;

(iii) $\mathfrak{T} \circ \mathfrak{T} = \mathfrak{T}$;

(iv) $\mathfrak{T}$ *is a group.*

PROOF. -

(i) $\implies$ (ii). By assumption for any $\tau_1, \tau_2 \in \mathfrak{T}$ and for any $x \in \mathcal{P}$, $\tau_1(x) = \tau_2(x)$ implies $\tau_1 = \tau_2$. For any $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \in \tilde{\mathcal{P}}$, if x is any point of $\mathcal{P}$, let us denote $\tilde{c}$ the reflection, uniquely determined by (A4) and (1.2) such that $\tilde{c}(\tilde{a}_3(x)) = \tilde{a}_1 \circ \tilde{a}_2(x)$. Then by assumption $\tilde{c} \circ \tilde{a}_3 = \tilde{a}_1 \circ \tilde{a}_2$ and hence by (A3) $\tilde{a}_1 \circ \tilde{a}_2 \circ \tilde{a}_3 = \tilde{c} \circ \tilde{a}_3 \circ \tilde{a}_3 = \tilde{c} \in \tilde{\mathcal{P}}$.

(ii) $\Longrightarrow$ (iii). Since $id \in \mathcal{T}$ we have: $\mathcal{T} \subseteq \mathcal{T} \circ \mathcal{T}$. If $\tilde{a}_1 \circ \tilde{b}_1$, $\tilde{a}_2 \circ \tilde{b}_2$ are any mappings of $\mathcal{T}$, by (ii) there exists $\tilde{c} \in \tilde{\mathcal{P}}$ such that $(\tilde{a}_1 \circ \tilde{b}_1) \circ$ $\circ (\tilde{a}_2 \circ \tilde{b}_2) = (\tilde{a}_1 \circ \tilde{b}_1 \circ \tilde{a}_2) \circ \tilde{b}_2 = \tilde{c} \circ \tilde{b}_2 \in \mathcal{T}$; thus $\mathcal{T} \circ \mathcal{T} \subseteq \mathcal{T}$.

(iii) $\Longrightarrow$ (iv). By (A3), for any $(\tilde{a} \circ \tilde{b}) \in \mathcal{T}$ we have $(\tilde{b} \circ \tilde{a}) = (\tilde{a} \circ \tilde{b})^{-1}$; thus by (iii) and since $id \in \mathcal{T}, \mathcal{T}$ is a group.

(iv) $\Longrightarrow$ (i). For any $\tau_1, \tau_2 \in \mathcal{T}$ and any $x \in \mathcal{P}$, $\tau_1(x) = \tau_2(x)$ implies $\tau_2^{-1} \circ \tau_1(x) = x$. Since by assumption $\tau_2^{-1} \circ \tau_1 \in \mathcal{T}$ and by (1.3, i) $\tau_2^{-1} \circ \tau_1 = id$, it follows $\tau_1 = \tau_2$ and thus because of the transitivity, $\mathcal{T}$ is regular on $\mathcal{P}$. □

1.5. - *If $\mathcal{T}$ is a group, then it is commutative.*

PROOF. - For any $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \in \tilde{\mathcal{P}}$, by (1.4, ii) and (A3), $\tilde{a}_1 \circ \tilde{a}_2 \circ \tilde{a}_3$ is involutory and then $\tilde{a}_1 \circ \tilde{a}_2 \circ \tilde{a}_3 = \tilde{a}_3 \circ \tilde{a}_2 \circ \tilde{a}_1$. Thus for any $\tilde{a}_1 \circ \tilde{b}_1$, $\tilde{a}_2 \circ \tilde{b}_2 \in \mathcal{T}$ we have $\tilde{a}_1 \circ \tilde{b}_1 \circ \tilde{a}_2 \circ \tilde{b}_2 = \tilde{a}_2 \circ \tilde{b}_1 \circ \tilde{a}_1 \circ \tilde{b}_2 = \tilde{a}_2 \circ \tilde{b}_2 \circ \tilde{a}_1 \circ \tilde{b}_1$; thus $\mathcal{T}$ is commutative. □

2. - The line structure

In $\mathcal{P} \times \mathcal{P}$ an *equivalence relation* Δ is defined fulfilling the following axioms:

D1. $\forall\, a, b, c \in \mathcal{P} : (a, b)\, \Delta\, (c, c) \iff a = b$;

D2. $\forall\, a, x, y \in \mathcal{P} : (x, y)\, \Delta\, (\tilde{a}(x), \tilde{a}(y))$;

D3. $\forall\, a, b, x \in P$, distinct: $(a, x)\, \Delta\, (x, b) \iff (a, x)\, \Delta\, (a, b)$.

2.1. - *For any $a, b, c \in \mathcal{P}$ we have:*

i) $(a, b)\, \Delta\, (b, a)$;

ii) $(a, c)\, \Delta\, (a, \tilde{a}(c))$;

iii) $\tilde{c}(a) = b \Longrightarrow (a, c)\, \Delta\, (c, b)$ *and* $(a, c)\, \Delta\, (a, b)$.

PROOF. -

(i) Let us denote $d \in \mathcal{P}$ the uniquely determined point, by (A4) and (1.2), such that $\tilde{d}(a) = b$ and then $\tilde{d}(b) = a$. Thus by (D2) $(a, b) \Delta (\tilde{d}(a), \tilde{d}(b)) = (b, a)$.

(ii) Follows from (D2) and (A1).

(iii) Follows from (D2), (i) and (D3). □

Let us define for any $a, b \in \mathcal{P}$, $a \neq b$:

(1) $$\overline{a, b} := \{x \in \mathcal{P} : (a, x) \Delta (a, b)\} \cup \{a\} .$$

Of course, by definition is $a, b \in \overline{a, b}$; furthermore we have:

2.2. - *For any* $a, b \in \mathcal{P}$, $a \neq b$, *it is:* $\overline{a, b} \setminus \{a, b\} \neq \emptyset$.

PROOF. - By (A4) it exists $c \neq a, b$ such that $\tilde{c}(a) = b$; then, by (2.1, iii): $(a, c) \Delta (a, b)$ that is $c \in \overline{a, b} \setminus \{a, b\}$. □

2.3. *For any* $a, b \in \mathcal{P}$, $a \neq b : \overline{a, b} = \overline{b, a}$.

PROOF. - By (D3), $\forall x \in \overline{a, b} \setminus \{a, b\} : (a, x) \Delta (a, b)$ implies $(a, x) \Delta (x, b)$; then by (2.1, i) and since Δ is transitive it is $(b, x) \Delta (b, a)$ and thus $x \in \overline{b, a}$. Furthermore $a, b \in \overline{b, a}$ by definition. Therefore $\overline{a, b} \subseteq \overline{b, a}$, and with the same arguments: $\overline{b, a} \subseteq \overline{a, b}$. □

2.4. - *For any* $c \in \overline{a, b} \setminus \{a\} : \overline{a, c} = \overline{a, b}$.

PROOF. - By definition $\forall x \in \overline{a, b} \setminus \{a\} : (a, x) \Delta (a, b) \Delta (a, c)$ implies $x \in \overline{a, c}$, $b \in \overline{a, c}$ i.e. $\overline{a, b} \subseteq \overline{a, c}$. With the same argument, since $b \in \overline{a, c} \setminus \{a\} : \overline{a, c} \subseteq \overline{a, b}$. □

2.5. - *For any* $c, e \in \overline{a, b}$, $c \neq e : \overline{c, e} = \overline{a, b}$.

PROOF. - If $c = a$, the theorem is proved by (2.4). $c \in \overline{a, b} \setminus \{a\}$ implies by (2.4) $\overline{a, b} = \overline{a, c}$; then again by (2.4) $e \in \overline{a, b} \setminus \{c\} = \overline{c, a} \setminus \{c\}$ implies $\overline{c, a} = \overline{c, e}$. □

The set of points $\overline{a, b}$ will be called *the line* $\overline{a, b}$. By (2.4) we have $|\overline{a, b}| \geqslant 3$.

The set of all lines defined by (1) will be denoted by $\mathcal{L}$ and thus the pair $(\mathcal{P}, \mathcal{L})$ turns out to be an *incidence space*. For any $a, b, c \in \mathcal{P}$, $a \neq b$ let us define:

$$\{c /\!\!/ \overline{a, b}\} := [x \in \mathcal{P} : (c, x) \, \Delta \, (a, b)] \cup \{c\} \tag{2}$$

2.6. - *$\{c /\!\!/ \overline{a, b}\}$ is a line and $\{c /\!\!/ \overline{a, b}\} = \overline{a, b}$ when $c \in \overline{a, b}$.*

PROOF. -

(i) If $c = a$ the definition (2) coincides with (1). If $c \in \overline{a, b} \setminus \{a\}$, by definition for any $x \in \{c /\!\!/ \overline{a, b}\} \setminus \{c\}$ we have $(a, c) \, \Delta \, (a, b) \, \Delta \, (c, x)$ and thus $(c, x) \, \Delta \, (c, a)$ which implies $\{c /\!\!/ \overline{a, b}\} = \overline{c, a} = \overline{a, b}$, because of (2.4).

(ii) Let us suppose $c \notin \overline{a, b}$; by (A4) there exists a point u such that $\tilde{u}(a) = c$. Then by (D2) $\forall x \in \{c /\!\!/ \overline{a, b}\} \setminus \{c\} : (c, x) \, \Delta \, (a, b) \, \Delta \, (c, \tilde{u}(b))$; hence by denoting $e := \tilde{u}(b)$ we have $(c, x) \, \Delta \, (a, b) \, \Delta \, (c, e)$ and thus by definition (1): $\{c /\!\!/ \overline{a, b}\} = \overline{c, e}$. □

Because of (2.2) and (2.6) we have proved that $|\{c /\!\!/ \overline{a, b}\}| \geqslant 3$; thus $\forall (a, b) \in \mathcal{P}^2$, $\forall c \in \mathcal{P}$ there exists at least one point e such that $(c, e) \, \Delta \, (a, b)$.

2.7. - *For any $e \in \{c /\!\!/ \overline{a, b}\}$ we have $\{e /\!\!/ \overline{a, b}\} = \{c /\!\!/ \overline{a, b}\}$.*

PROOF. - If $\{c /\!\!/ \overline{a, b}\} = \overline{a, b}$ by (2.6) we have $\{e /\!\!/ \overline{a, b}\} = \overline{a, b}$. If $e = c$ the statement is trivial. Then let us suppose $e \neq c$ and $c \notin \overline{a, b}$. By definition $e \in \{c /\!\!/ \overline{a, b}\} \setminus \{c\}$ implies $(c, e) \, \Delta \, (a, b)$; then $\forall x \in \{c /\!\!/ \overline{a, b}\} \setminus \setminus \{c\} : (c, x) \, \Delta \, (a, b) \, \Delta \, (c, e)$ and by (D3) if $x \neq e$ $(c, x) \, \Delta \, (c, e)$ implies $(c, x) \, \Delta \, (x, e)$. Thus $(e, x) \, \Delta \, (a, b)$ and hence $x \in \{e /\!\!/ \overline{a, b}\}$, i.e. $\{c /\!\!/ \overline{a, b}\} \subseteq \subseteq \{e /\!\!/ \overline{a, b}\}$.

Since $c \in \{e /\!\!/ \overline{a, b}\}$, with the same arguments we have also $\{e /\!\!/ \overline{a, b}\} \subseteq \{c /\!\!/ \overline{a, b}\}$, and thus $\{e /\!\!/ \overline{a, b}\} = \{c /\!\!/ \overline{a, b}\}$.

REMARK I. - Because of (2.7) we have, for any $a, b, c, d \in \mathcal{P}$, with $a \neq b : \{c /\!\!/ \overline{a, b}\} \cap \{d /\!\!/ \overline{a, b}\} \neq \emptyset$ implies $\{c /\!\!/ \overline{a, b}\} = \{d /\!\!/ \overline{a, b}\}$; in fact if $e \in \{C /\!\!/ \overline{a, b}\} \cap \{d /\!\!/ \overline{a, b}\}$, by (2.7) we have: $\{c /\!\!/ \overline{a, b}\} = \{e /\!\!/ \overline{a, b}\} = \{d /\!\!/ \overline{a, b}\}$.

For any two lines $\overline{a, b}, \overline{c, d}, \in \mathcal{L}$ let us now define:

$$\overline{a, b} \mathbin{/\!\!/} \overline{c, d} \Longleftrightarrow (a, b) \, \Delta \, (c, d) .$$

Since Δ is an equivalence relation, $/\!\!/$ is also an equivalence relation. Furthermore, because of (2.6), for any $\overline{a, b} \in \mathcal{L}$ and any $c \in \mathcal{P}$ there exists a line $L \in \mathcal{L}$ such that $c \in L$ and $L \mathbin{/\!\!/} \overline{a, b}$. Actually we have $L := \{c \mathbin{/\!\!/} \overline{a, b}\}$. Moreover by (2.7) this line L is unique. Thus "$/\!\!/$" is an *equivalence relation* defined in $\mathcal{L}$ which fulfils the Euclidean axiom; "$/\!\!/$" will be called *parallelism* and the triple $(\mathcal{P}, \mathcal{L}, /\!\!/)$ turns out to be an *incidence space with parallelism.*

3. - The reflection geometry

We can now study some properties of an incidence space with parallelism endowed with a set of reflections as defined in § 1.

3.1. - *For any $L \in \mathcal{L}$ and $a \in \mathcal{P}$ we have:*

(i) $\tilde{a}(L) \in \mathcal{L}$ *and* $\tilde{a}(L) \mathbin{/\!\!/} L$;

(ii) $\tilde{a}(L) = L \Longleftrightarrow a \in L$.

PROOF. -

(i) Let be $L =: \overline{u, v} = [x \in \mathcal{P} : (u, x) \, \Delta \, (u, v)] \cup \{u\}$; then $\tilde{a}(L) = [\tilde{a}(x) : (u, x) \, \Delta \, (u, v)] \cup \{\tilde{a}(u)\}$ and since by (D2) $(u, x) \, \Delta \, (\tilde{a}(u), \tilde{a}(x))$, by denoting $y := \tilde{a}(x)$ we have by (2) : $\tilde{a}(L) = \{y \in \mathcal{P} : (\tilde{a}(u), y) \, \Delta \, (u, v)\} \cup \cup \{\tilde{a}(u)\} = \{\tilde{a}(u) \mathbin{/\!\!/} L\} \in \mathcal{L}$.

(ii) "$\Longleftarrow$" If $a \in L =: \overline{u, v}$ $(u, a) \, \Delta \, (u, v)$ implies by (D2) $(\tilde{a}(u), a) \, \Delta \, \Delta \, (u, v)$ i.e. $a \in [\tilde{a}(u) \mathbin{/\!\!/} \overline{u, v}] = \tilde{a}(L)$ and thus $\tilde{a}(L) = L$ because of the Remark I, § 2.

"$\Longrightarrow$" Let $L =: \overline{u, v}$; by (D2) $(u, a) \, \Delta \, (a, u) \, \Delta \, (a, \tilde{a}(u))$ which implies, by (D3): $(u, a) \, \Delta \, (u, \tilde{a}(u))$. Furthermore $\tilde{a}(L) = |\tilde{a}(u) \mathbin{/\!\!/} L| = L = \overline{u, v}$ implies $\tilde{a}(u) \in \overline{u, v}$ and thus $(u, \tilde{a}(u)) \, \Delta \, (u, v)$. Hence we have $(u, a) \, \Delta \, (u, v)$ which means $a \in \overline{u, v}$. □

By (3.1) we have proved the following situation.

$(\mathcal{P}, \mathcal{L}, /\!/)$ is an *incidence space with parallelism* where the set of reflections $\tilde{\mathcal{P}} := \{\tilde{a} : a \in \mathcal{P}\}$ is a *set of involutory dilatations*, acting *transitively on* $\mathcal{P}$ because of (A4).

Such an incidence space with the set $\tilde{\mathcal{P}}$ of dilatations fulfilling the axioms (A1)-(A4) will be denoted henceforth by $(\mathcal{P}, \mathcal{L}, /\!/, \tilde{\mathcal{P}})$ and will be called a *point-reflection geometry*.

Furthermore by (1.3) $\mathcal{T} := \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} = \{\tilde{a} \circ \tilde{b} : a, b \in \mathcal{P}\}$ is a *set of translations* acting *transitively on* $\mathcal{P}$. Henceforth we shall denote by $\mathfrak{D} := Aut(\mathcal{P}, \mathcal{L}, /\!/)$ the *group* of all *dilatations* of $(\mathcal{P}, \mathcal{L}, /\!/)$ and by $\mathcal{K}$ the *set* of all *translations*, i.e. fixed-point-free dilatations, of $(\mathcal{P}, \mathcal{L}, /\!/)$ together with the identity map "*id*".

Now, by denoting $u \in \mathcal{P}$ a distinguished point, let us define

$$\mathcal{C} := \tilde{\mathcal{P}} \circ \tilde{u} = \{\tilde{a} \circ \tilde{u} : a \in \mathcal{P}\} .$$

Thus $\mathcal{C}$ is a *set of translations* acting *regularly on* $\mathcal{P}$. In fact $\forall\, x, y \in \mathcal{P}$, since by (A4) it exists $c \in \mathcal{P}$ such that $\tilde{c}(y) = \tilde{u}(x)$ we have $\tilde{c} \cdot \tilde{u}(x) = y$ and thus $\mathcal{C}$ is transitive.

Furthermore if $\tilde{c} \circ \tilde{u}(x) = \tilde{e} \circ \tilde{u}(x)$ we have, by denoting $z := \tilde{u}(x)$, $\tilde{c}(z) = \tilde{e}(z)$ which implies, by (1.1, ii) $\tilde{c} = \tilde{e}$. Furthermore we have: $id \in \mathcal{C}$.

3.2. - *The set* $\mathcal{T}$ *is a group if and only if* $\mathcal{C} = \mathcal{T}$.

PROOF. - "$\Longrightarrow$" By definition $\mathcal{C} \subseteq \mathcal{T}$. On the other hand for any $\tilde{a} \circ \tilde{b} \in \mathcal{T}$, if $c \in \mathcal{P}$ is the point, uniquely determined by (A4), such that $\tilde{c}(u) = \tilde{a} \circ \tilde{b}(u)$, we have by (1.4, i): $\tilde{a} \circ \tilde{b} = \tilde{c} \circ \tilde{u}$ and then $\mathcal{T} \subseteq \mathcal{C}$.

"$\Longleftarrow$" By definition $\mathcal{C}$ is acting regularly on $\mathcal{P}$ and thus by (1.4) $\mathcal{T}$ is a group. □

3.3. - *In* $(\mathcal{P}, \mathcal{L}, /\!/, \tilde{\mathcal{P}})$ *the followings hold:*

i) *for any three non collinear points* a, b, c *the parallelogram configuration holds, i.e.:* $\{b /\!/ \overline{a, c}\} \cap \{c /\!/ \overline{a, b}\} \neq \emptyset$*; furthermore in any parallelogram* (a, b, c, x) *the diagonal lines do meet;*

ii) *if* $\delta \in Aut(\mathcal{P}, \mathcal{L}, /\!/) \setminus \{id\}$ *with* $\delta^2 = id$*, then* $\delta \in \tilde{\mathcal{P}}$.

PROOF. -

(i) Let $u \in \mathcal{P}$ be the point, uniquely determined by (A4), such that $\tilde{u}(b) = c$. Then $\tilde{u}(\overline{a, c}) = \{b /\!/ \overline{a, c}\}$, $\tilde{u}(\overline{a, b}) = \{c /\!/ \overline{a, b}\}$ and so $\{x\} := \{\tilde{u}(a)\} = \{b /\!/ \overline{a, c}\} \cap \{c /\!/ \overline{a, b}\} \neq \emptyset$ since $\tilde{u}$ is a bijection on $\mathcal{P}$. Thus $\{u\} = \overline{a, x} \cap \overline{b, c}$.

(ii) Let $a \in \mathcal{P}$ be any point with $\delta(a) \neq a$ and $b := \delta(a)$. For any $x \in \mathcal{P} \setminus \overline{a, b}$ we have $\delta(x) := \delta(\overline{a, x} \cap \overline{b, x}) = \{b /\!/ \overline{a, x}\} \cap \{a /\!/ \overline{b, x}\}$, and $\delta(x)$ does exist since δ is a bijection. If $u \in \mathcal{P}$ is the point, uniquely determined by (A4), such that $\tilde{u}(a) = b$, by (i) we have $\tilde{u}(x) = \delta(x)$ and thus, since δ and $\tilde{u}$ are dilatations, $\delta = \tilde{u}$. □

REMARK I. - As a consequence of (3.3, ii), in $(\mathcal{P}, \mathcal{L}, /\!/, \tilde{\mathcal{P}})$ do not exist involutory translations.

Because of (3.3, i) the following configurational proposition (*T*) holds in any point-reflection geometry $(\mathcal{P}, \mathcal{L}, /\!/, \tilde{\mathcal{P}})$.

3.4. - *For any three non collinear points* $a_1, a_2, a_3 \in \mathcal{P}$ *there exist three non collinear points* b_1, b_2, b_3 *such that (configuration* ***T****):*

$\{b_3\} := \{a_1 /\!/ \overline{a_2, a_3}\} \cap \{a_2 /\!/ \overline{a_1, a_3}\} \neq \emptyset;$

$\{b_1\} := \{a_2 /\!/ \overline{a_1, a_3}\} \cap \{a_3 /\!/ \overline{a_1, a_2}\} \neq \emptyset;$

$\{b_2\} := \{a_3 /\!/ \overline{a_1, a_2}\} \cap \{a_1 /\!/ \overline{a_2, a_3}\} \neq \emptyset.$

The triangle Tr (a_1, a_2, a_3) is said to be *inscribed* in Tr (b_1, b_2, b_3) and furthermore Tr (a_1, a_2, a_3) and Tr (b_1, b_2, b_3) are said to be *similar*. By (3.4) we know that any triangle Tr (a_1, a_2, a_3) can be inscribed in a similar one but we don't know if, vice versa, any triangle Tr (b_1, b_2, b_3) can circumscribe a similar one.

By definition we know that $\mathcal{T} := \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \subseteq \mathcal{K} \subseteq \mathcal{D}$ and, since $id \in \mathcal{T}$, $\tilde{\mathcal{P}} \subseteq \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \subseteq \mathcal{D}$. What in general we don't know is whether (for any $a, b, c \in \mathcal{P}$) the dilatation $\tilde{a} \circ \tilde{b} \circ \tilde{c}$ has one fixed point or not. In other words it is not known whether in a general point-reflection geometry $(\mathcal{P}, \mathcal{L}, /\!/, \tilde{\mathcal{P}})$ it is $\tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \cap \mathcal{K} = \emptyset$ or not. Furthermore if $\tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \cap \mathcal{K} = \emptyset$, the proper dilatation $\tilde{a} \circ \tilde{b} \circ \tilde{c}$ (for any $a, b, c \in \mathcal{P}$) can be involutory or not.

When $\tilde{a} \circ \tilde{b} \circ \tilde{c}$ is involutory (for any $a, b, c \in \mathcal{P}$) we have by (3.3, ii): $\tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \subseteq \tilde{\mathcal{P}}$ and thus $\tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} \circ \tilde{\mathcal{P}} = \tilde{\mathcal{P}}$. In this case by (1.4) $\mathcal{T}$ is a commutative group.

3.5. *Let $a_1, a_2, a_3 \in \mathcal{P}$ be non collinear and such that $\delta := \tilde{a}_1 \circ \tilde{a}_2 \circ \tilde{a}_3$ is a proper dilatation; let us denote by x the fixed points of δ. Then three points $b_1, b_2, b_3 \in \mathcal{P}$ are uniquely determined such that, for any permutation (i, j, h) of $(1, 2, 3)$, it holds:*

$$a_i \in \overline{b_j, b_h}\,, \qquad b_h := \tilde{a}_i (b_j)\,, \qquad b_2 := x\,;$$

this implies also:

$$\tilde{a}_i \circ \tilde{a}_j \circ \tilde{a}_h (b_j) = b_j\,.$$

PROOF. - $\tilde{a}_1 \circ \tilde{a}_2 \circ \tilde{a}_3 (x) = x$ implies $\tilde{a}_2 \circ \tilde{a}_3 (x) = \tilde{a}_1 (x)$; thus by denoting $b_1 := \tilde{a}_3 (x)$, $b_2 := x$, $b_3 := \tilde{a}_1 (x)$ we have $\tilde{a}_2 (b_1) = b_3$ and, by (D2) and the definition (1) of line, $a_i \in \overline{b_j, b_h}$. □

REMARK II. - The triangle Tr (a_1, a_2, a_3) defined in (3.5) is inscribed in Tr (b_1, b_2, b_3); Tr (b_1, b_2, b_3) will be said the *fixed-points-triangle* for the triple of reflections $\{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3\}$. Now by (3.3) let us denote by $\{c_i\} := \{b_j /\!/ \overline{b_i, b_h}\} \cap \{b_h /\!/ \overline{b_i, b_j}\}$; then the triangle Tr (c_1, c_2, c_3) circumscribes the similar triangle Tr (b_1, b_2, b_3). Hence, since by (3.3, i) $\tilde{a}_i (b_j) = b_h$ implies also $\tilde{a}_i (c_i) = b_i$, we have:

$$\tilde{a}_i \circ \tilde{a}_j \circ \tilde{a}_h (c_h) = c_i\,.$$

Using the same notations of (3.5) and Remark II, we can now state the following propositions. Actually (3.6) and (3.7) follow immediately from (3.5) and (A4).

3.6. - *For any $a_1, a_3, x \in \mathcal{P}$ non collinear, one point $a_2 \in \mathcal{P}$ is uniquely determined such that $\tilde{a}_1 \circ \tilde{a}_2 \circ \tilde{a}_3 (x) = x$.*

3.7. - *For any $b_1, b_2, b_3 \in \mathcal{P}$ non collinear, a triple of reflections $\{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3\}$ admitting* Tr (b_1, b_2, b_3) *as the fixed-points-triangle is uniquely determined.*

3.8. - *For any* $b_1, b_2, b_3 \in \mathcal{P}$ *non collinear, if* Tr (b_1, b_2, b_3) *is the fixed-points-triangle for a triple of reflections* $[\tilde{a}_1, \tilde{a}_2, \tilde{a}_3]$, Tr (a_1, a_2, a_3) *is inscribed and similar to* Tr (b_1, b_2, b_3) *if and only if* $\mathcal{T}$ *is a group.*

PROOF. - " $\Longleftarrow$ " By (1.5) the group $\mathcal{T}$ is commutative. In this case we know (cfr. [1]) that, for any $\tau \in \mathcal{T}$ and for any $x \in \mathcal{P}$, $L \in \mathcal{L}$, $L \mathbin{/\!/} \overline{x, \tau(x)}$ implies $\tau(L) = L$. With the notations of (3.5) we have $\tilde{a}_i \circ \tilde{a}_h(b_i) = b_h$ and $\tilde{a}_i \circ \tilde{a}_h \in \mathcal{T}$; hence $\tilde{a}_i \circ \tilde{a}_h(\overline{b_i, b_h}) = \{b_h \mathbin{/\!/} \overline{b_i, b_h}\} = \overline{b_i, b_h}$. Otherwise $\tilde{a}_i \circ \tilde{a}_h(a_h) = \tilde{a}_i(a_h) \in \overline{a_i, a_h}$ implies $\tilde{a}_i \circ \tilde{a}_h(\overline{a_i, a_h}) = \overline{a_i, a_h}$. Thus $\overline{b_i, b_h} \mathbin{/\!/} \overline{a_i, a_h}$.

" $\Longrightarrow$ " Again with the notations of (3.5) and Remark II we have: $\tilde{a}_h \circ \tilde{a}_i(b_h) = b_i$ and $\tilde{a}_h \circ \tilde{a}_i(b_j) = c_h$. By assumption b_1, b_2, b_3 are any three non collinear points and $\tau := \tilde{a}_h \circ \tilde{a}_i \in \mathcal{T}$ is a translation of $\mathcal{T}$ mapping b_h onto b_i and b_j onto c_h. Then τ fulfils the law of parallelograms i.e. $\{\tau(b_j)\} = \{b_j \mathbin{/\!/} \overline{b_i, b_h}\} \cap \{b_i \mathbin{/\!/} \overline{b_j, b_h}\}$ and then acts regularly on $\mathcal{P}$; this implies by (1.4) that $\mathcal{T}$ is a group. □

Let now L be any line. We shall denote $\tilde{L} := \{\tilde{a} \in \tilde{\mathcal{P}} : a \in L\}$, $\mathcal{T}_L := \tilde{L} \circ \tilde{L} = \{\tilde{a} \circ \tilde{b} : a, b \in L\}$.

By (3.1) we have $\tilde{L}(L) = L$ and $\mathcal{T}_L(L) = L$. Thus $\tilde{L} \circ \tilde{L} \circ \tilde{L}$ is a set of dilatations with the possible fixed point on L.

By (1.4), since the axioms (A1)-(A4) are independent from the axioms of the line structure we have:

3.9. - *Let* L *be any line of a point-reflection geometry* $(\mathcal{P}, \mathcal{L}, \mathbin{/\!/}, \tilde{\mathcal{P}})$. *Then the following conditions are equivalent:*

i) $\mathcal{T}_L$ *acts regularly on* L;

ii) $\tilde{L} \circ \tilde{L} \circ \tilde{L} = \tilde{L}$;

iii) $\mathcal{T}_L$ *is a group.*

4. - The associated incidence loop

It is well known (see e.g. [5]) that an incidence space with parallelism $(\mathcal{P}, \mathcal{L}, \mathbin{/\!/})$ together with a regular set of translations $\mathcal{S}$ can be represented as an incidence loop in the following way.

If 1 is a distinguished point of $\mathcal{P}$, for any point $a \in \mathcal{P}$ let us denote $a^{\bullet}$ the uniquely determined translation $a^{\bullet} \in \mathcal{S}$ such that $a^{\bullet}(1) = a$. Then we can define in $\mathcal{P}$ a composition low "$\bullet$" by denoting for any $a, b \in \mathcal{P}$, $a \bullet b := a^{\bullet}(b) = a^{\bullet} \circ b^{\bullet}(1)$; in this way $(\mathcal{P}, \mathcal{L}, /\!\!/, \bullet)$ turns out to be an *incidence loop with parallelism* (cf. [3] and [5]). In the present situation, we can assume $\mathcal{S} := \mathcal{C}$ and for the sake of semplicity $1 := u$. As a consequence for any $a \in \mathcal{P}$, if we denote $a_m \in \mathcal{P}$ the point, uniquely determined by (A4) and (1.2), such that $\tilde{a}_m(u) = a$, we shall have:

$$a^{\bullet} := \tilde{a}_m \circ \tilde{u}\,; \tag{3}$$

and, for any $a, b \in \mathcal{P}$:

$$a \bullet b := a^{\bullet}(b) = a^{\bullet} \circ b^{\bullet}(u) = \tilde{a}_m \circ \tilde{u} \circ \tilde{b}_m(u) = (\tilde{a}_m \circ \tilde{u})(b)\,. \tag{4}$$

Then with the usual notations (cfr. [3]) we shall represent by $(\mathcal{P}, \bullet)$ the set $\mathcal{P}$ of points endowed with the loop structure defined in (3), by $\mathcal{L} := \{aL : a \in \mathcal{P}, u \in L \in \mathcal{L}\}$ the set of lines and by the condition:

$$aL /\!\!/ bM \Longleftrightarrow L = M$$

the parallelism relation between nay two lines $aL, bM \in \mathcal{L}$. Furthermore the left multiplication in $(\mathcal{P}, \bullet)$ will represent the translations of the set $\mathcal{C}$. The incidence loop with parallelism $(\mathcal{P}, \mathcal{L}, /\!\!/, \bullet)$ defined in this way from the reflection geometry $(\mathcal{P}, \mathcal{L}, /\!\!/, \tilde{\mathcal{P}})$ will be called *incidence loop with reflections* and will be denoted by $(\mathcal{P}, \mathcal{L}, /\!\!/, \bullet, \sim)$.

4.1. - *Let* $(\mathcal{P}, \mathcal{L}, /\!\!/, \bullet, \sim)$ *be an incidence loop with reflections as previously defined. The following properties hold.*

i) $\mathcal{C} \circ \mathcal{C} \subseteq \mathcal{T}$;

ii) $\forall\, a \in \mathcal{P}\ \exists\, a^{-1} \in \mathcal{P}$ *such that:* $a^{-1}a = aa^{-1} = u$;

iii) $\forall\, a \in \mathcal{P} : (a^{\bullet})^{-1} = (a^{-1})^{\bullet}$; *then* $\mathcal{C}^{-1} = \mathcal{C}$;

iv) $\forall\, a \in \mathcal{P} : \tilde{u}(a) = a^{-1}$;

v) $\forall\, a \in \mathcal{P} : \tilde{a}(u) = a^2$.

PROOF. -

(i) Let be $\tilde{a} \circ \tilde{u}$, $\tilde{b} \circ \tilde{u} \in \mathcal{C}$. Since $(\tilde{u} \circ \tilde{b} \circ \tilde{u})(\tilde{u} \circ \tilde{b} \circ \tilde{u}) = id$, the dilatation $\tilde{u} \circ \tilde{b} \circ \tilde{u}$ is involutory and thus by (3.3, ii) it exists $c \in \mathcal{P}$ with $\tilde{c} := \tilde{u} \circ \tilde{b} \circ \tilde{u}$; this implies $\tilde{a} \circ \tilde{u} \circ \tilde{b} \circ \tilde{u} = \tilde{a} \circ \tilde{c} \in \mathfrak{T}$.

(ii) By [6] and [7] it will be enough to prove that, for any $\alpha, \beta \in \mathcal{C}$, $\alpha \circ \beta(u) = u$ implies $\beta \circ \alpha(u) = u$. Actually let us denote $\alpha := \tilde{a}_m \circ \tilde{u}$, $\beta := \tilde{b}_m \circ \tilde{u}$; then $u = \alpha \circ \beta(u) = \tilde{a}_m \circ \tilde{u} \circ \tilde{b}_m \circ \tilde{u}(u) = \tilde{a}_m \circ \tilde{u} \circ \tilde{b}_m(u)$ implies $\tilde{b}_m \circ \tilde{u} \circ a_m(u) = u$. Then $\beta \circ \alpha(u) = \tilde{b}_m \circ \tilde{u} \circ \tilde{a}_m \circ \tilde{u}(u) = \tilde{b}_m \circ \tilde{u} \circ \tilde{a}_m(u) = u$ and thus by [6] (proposition 1.2) or [7] (proposition 4, § 2) it exists a^{-1}.

(iii) By definition and by (ii) we have $a \bullet a^{-1} = a^\bullet(a^{-1}) = u$ which implies $a^{-1} = (a^\bullet)^{-1}(u)$. Otherwise, again by definition, $a^{-1} = (a^{-1})^\bullet(u)$; then $(a^\bullet)^{-1}(u) = (a^{-1})^\bullet(u)$ that is $a^\bullet \circ (a^{-1})^\bullet(u) = u$. By (i) $a^\bullet \circ (a^{-1})^\bullet \in \mathfrak{T}$ which implies $a^\bullet \circ (a^{-1})^\bullet = id$, i.e. $(a^{-1})^\bullet = (a^\bullet)^{-1}$. By (ii) $\mathcal{P}^{-1} = \mathcal{P}$ and thus $\mathcal{C}^{-1} = \mathcal{C}$.

(iv) Since $\tilde{u}, a^\bullet \in \mathfrak{D}$, $\tilde{u}(a) = \tilde{u} \circ a^\bullet(u) = \tilde{u} \circ \tilde{a}_m \circ \tilde{u}(u)$. Furthermore $\tilde{u} \circ \tilde{a}_m = (\tilde{a}_m \circ \tilde{u})^{-1} = (a^\bullet)^{-1} = (a^{-1})^\bullet$. Then: $\tilde{u}(a) = (a^{-1})^\bullet \circ \tilde{u}(u) = a^{-1}$.

(v) Let us consider the dilatation $\delta := a^\bullet \circ \tilde{u} \circ (a^\bullet)^{-1}$; since $\delta(a) = a^\bullet \circ \tilde{u} \circ (a^\bullet)^{-1}(a) = a^\bullet \circ \tilde{u}(u) = a$ and $\delta \circ \delta = id$, by (A1), (A2) and (3.3, ii) we have $\delta = \tilde{a}$. Then by (iii) and (iv) $\tilde{a}(u) = \delta(u) =$ $= a^\bullet \circ \tilde{u} \circ (a^\bullet)^{-1}(u) = a^\bullet \circ \tilde{u} \circ (a^{-1})^\bullet(u) = a^\bullet \circ \tilde{u}(a^{-1}) = a^\bullet(a) = a^2$.

□

REMARK I. - In (4.1, i) we have proved that, for any $\tilde{b} \in \tilde{\mathcal{P}}$ we have $\tilde{u} \circ \tilde{b} \circ \tilde{u} \in \tilde{\mathcal{P}}$. By denoting $\tilde{c} := \tilde{u} \circ \tilde{b} \circ \tilde{u}$ we obtain $c = \tilde{c}(c) =$ $= \tilde{u} \circ \tilde{b} \circ \tilde{u}(c)$, i.e. $\tilde{u}(c) = \tilde{b} \circ \tilde{u}(c)$. Then, by (A1) and (A2) $\tilde{u}(c) = b$, and by (4.1, iv) $c = b^{-1}$. Hence $(\tilde{a} \circ \tilde{u}) \circ (\tilde{b} \circ \tilde{u}) = \tilde{a} \circ \tilde{c} = \tilde{a} \circ (\widetilde{b^{-1}})$.

REMARK II. - By (4.1, v) we know that, for any $b \in \mathcal{P}$, $\tilde{b}(u) = b^2$. By (A4) and (1.2), for any $a \in \mathcal{P}$, it exists a unique $a_m \in \mathcal{P}$ such that $\tilde{a}_m(u) = a$ and in this way we have defined $a^\bullet$ by (3). Then by denoting $b := a_m$ we can see that for any $a \in \mathcal{P}$ it exists the square root "b" i.e. $b \in \mathcal{P}$ such that $b^2 = a$.

4.2. - *Let* $(\mathcal{P}, \mathcal{L}, /\!\!/, \bullet)$ *be an incidence loop with parallelism fulfilling the following conditions*:

i) *if* 1 *denotes the unitary element of* $(\mathcal{P}, \bullet)$, *for any* $a \in \mathcal{P}$ *there exists* $a^{-1} \in \mathcal{P}$ *such that* $a \bullet a^{-1} = a^{-1} \bullet a = 1$;

ii) *there exists a dilatation* $\omega \in Aut(\mathcal{P}, \mathcal{L}, /\!/) \setminus [id]$ *such that* $\omega(1) = 1$, $\omega^2 = id$ *and, for any* $a \in \mathcal{P}$, $\omega(a) = a^{-1}$;

iii) *for any* $x, y \in \mathcal{P}$ *there exists a unique solution "a" for the equation*

$$(a^{-1}x)\ (a^{-1}y) = 1\,. \tag{5}$$

Then a set of reflections $\wedge : \mathcal{P} \longrightarrow \mathcal{P}$; $a \longrightarrow \hat{a}$ *can be defined fulfilling the axioms* (A1)-(A4) *such that* $(\mathcal{P}, \mathcal{L}, /\!/, \hat{\mathcal{P}})$ *is a point-reflection geometry.*

PROOF. - For any $a \in \mathcal{P}$ let us define: $\hat{a} := a^\bullet \circ \omega\, (a^\bullet)^{-1}$; since $a^\bullet$ is a translation for $(\mathcal{P}, \mathcal{L}, /\!/)$, $(a^\bullet)^{-1} \in Aut(\mathcal{P}, \mathcal{L}, /\!/)$ and thus $\hat{a} \in Aut(\mathcal{P}, \mathcal{L}, /\!/)$. Since by definition $a = a^\bullet(1)$ implies $(a^\bullet)^{-1}(a) = 1$, we have $\hat{a}(a) = a^\bullet \circ \omega \circ (a^\bullet)^{-1}(a) = a^\bullet \circ \omega\,(1) = a$ and thus (A1) is fulfilled. Furthermore by $\hat{a} \circ \hat{a} = a^\bullet \circ \omega \circ (a^\bullet)^{-1} \circ a^\bullet \circ \omega \circ (a^\bullet)^{-1} = id$, (A3) is fulfilled too. Since $1^\bullet = id$ we have also $\omega = \hat{1}$. In order to prove the axioms (A2) and (A4) we have to compare $(a^\bullet)^{-1}$ and $(a^{-1})^\bullet$. By the definition of $a^\bullet$ we have $(a^\bullet)^{-1}(a) = 1$ and, by (i), $a^\bullet(a^{-1}) = 1$ implies $(a^\bullet)^{-1}(1) = a^{-1}$; furthermore by (i) we have $(a^{-1})^\bullet(a) = 1$ and, by definition, $(a^{-1})^\bullet(1) = a^{-1}$. Thus since $(a^\bullet)^{-1}, (a^{-1})^\bullet \in Aut(\mathcal{P}, \mathcal{L}, /\!/)$ we have $(a^\bullet)^{-1} = (a^{-1})^\bullet$.

Let us now consider, for any $x, y \in \mathcal{P}$ the condition $\hat{a}(x) = y$: this is equivalent to $a^\bullet \circ \omega \circ (a^\bullet)^{-1}(x) = y$ i.e. $\omega \circ (a^\bullet)^{-1}(x) = (a^\bullet)^{-1}(y)$. By the properties of ω and $(a^\bullet)^{-1}$ we have $(a^\bullet)^{-1}(y) = (a^{-1})^\bullet(y) = a^{-1}y$ and $\omega \circ (a^\bullet)^{-1}(x) = (a^{-1}x)^{-1}$. Then $\hat{a}(x) = y$ gives rise to the equation (5) which has a unique solution $a \in \mathcal{P}$; thus (A2) and (A4) are fulfilled. Furthermore, since $\hat{\mathcal{P}} := [\hat{a} : a \in \mathcal{P}] \subseteq Aut(\mathcal{P}, \mathcal{L}, /\!/)$, the axiom (D2) is fulfilled, while (D3) is a consequence of the existence of lines and parallelism. □

REMARK III. - Let us suppose in the loop $(\mathcal{P}, \bullet)$ the property L.I.P. (*left inverse property*; cf. [7]) holds, i.e. for any $a, b \in \mathcal{P}$: $a^{-1}(a \bullet b) = (a^{-1} \bullet a)\, b$.

Then the existence of solutions for the equation (5) implies the existence of square roots for any element of $(\mathcal{P}, \bullet)$. Actually by assuming $x := 1$, for any $y \in \mathcal{P}$, the solution "a" of (5) is such that $a^{-1}[a^{-1}(a \bullet a)] = a^{-1}[(a^{-1} \bullet a) \bullet a] = 1$, i.e. $y = a^2$.

REMARK IV. - Let us now suppose that incidence loop $(\mathcal{P}, \mathcal{L}, /\!/, \cdot)$ considered in (4.2) is provided as well with a set of reflections $\tilde{\mathcal{P}}$ which means it is an *incidence loop with reflections* $(\mathcal{P}, \mathcal{L}, /\!/, \cdot, \sim)$. By (3.3, ii) we know that, for any $a \in \mathcal{P}$, $\hat{a} \in \tilde{\mathcal{P}}$; thus, since $\hat{a}(a) = a$, by (A2) we have $\hat{a} = \tilde{a}$; hence $(\mathcal{P}, \mathcal{L}, /\!/, \cdot, \sim)$ and $(\mathcal{P}, \mathcal{L}, /\!/, \cdot, \wedge)$ are isomorphic.

REFERENCES

[1] J. ANDRÈ, *Über Parallelstrukturen*, II. Math. Z.; 76 (1961), 155-163.

[2] F. BACHMANN, *Aufbau der Geometrie aus dem Spiegelungsbegriff*, Springer Verlag, Berlin, Göttingen, Heidelberg, 1959.

[3] H. KARZEL, G.P. KIST, *Kinematic Algebras and their Geometries*, Ring and Geometry (Kaya et al. editors). Nato ASI Series C, vol. 160 (1985), 437-509.

[4] H. KARZEL, K. SÖRENSEN, D. WINDELBERG, *Einführung in die Geometrie*, Vandenhoeck & Ruprecht, Göttingen, 1973.

[5] M. MARCHI, *Configurations in Incidence Loops*, J.C.I.S.S., 15 (1990), 287-300.

[6] M. MARCHI, *Incidence Loops and their Geometry*, COMBINATORICS '90; Proceedings of the conference on Combinatorics, Gaeta, Italy, 20-27 May, 1990; (Barlotti et al. editors). North Holland (1992), 347-364.

[7] E. ZIZIOLI, *Fibered Incidence Loops and Kinematic Loops*, Journal of Geometry, 30 (1987), 144-156.